Fermi National Accelerator Laboratory

FERMILAB–Pub–94/175-A
astro-ph/9407037
submitted to *Physical Review Letters*

Detectability of Tensor Perturbations Through CBR Anisotropy

Lloyd Knox[1,2] and Michael S. Turner[1,2,3]

[1] *NASA/Fermilab Astrophysics Center*
Fermi National Accelerator Laboratory, Batavia, IL 60510-0500

[2] *Department of Physics*
The University of Chicago, Chicago, IL 60637-1433

[3] *Department of Astronomy & Astrophysics*
Enrico Fermi Institute, The University of Chicago, Chicago, IL 60637-1433

ABSTRACT

Detection of tensor perturbations predicted in inflationary models provides an important test of inflation as well as crucial data for reconstructing the inflationary potential. We show that T/S must be greater than 0.2 for a statistically significant detection of tensor perturbations—even with perfect knowledge of the CBR sky temperature (T, S are respectively the tensor and scalar contribution to the variance of the CBR quadrupole anisotropy). This sensitivity can be achieved by full-sky measurements at the 3° and 0.5° angular scales.

Inflation is a very attractive early-Universe paradigm, and has had almost as much impact on cosmology as the big-bang model itself. Inflation provides not only an explanation for the flatness and smoothness of the observed Universe, but also for the inhomogeneity needed to seed all the structure seen today. However, inflation has yet to be tested in a significant way.

The key to testing inflation are its three robust predictions: spatially-flat Universe ($\Omega_0 = 1$) and nearly scale-invariant spectra of scalar (density) [1] and tensor (gravity-wave) [2] metric perturbations. With regard to the first, there is some evidence that the density parameter Ω_0 is close to unity [3], and several large-scale experiments are underway to search for the nonbaryonic dark matter that must be present if $\Omega_0 = 1$. Scalar perturbations seed the formation of structure, and hence, a large body of observational data concerning the distribution of matter in the Universe today can provide information about them. Today, tensor perturbations correspond to a stochastic background of gravitational waves that could possibly be detected by space-based gravity-wave detectors. Both tensor and scalar metric perturbations give rise to anisotropy in the temperature of the Cosmic Background Radiation (CBR) on angular scales from arcminutes to 180° (see Fig. 1), and CBR-anisotropy experiments are a promising means of testing these predictions of inflation.

Long before inflation the attractiveness of scale-invariant perturbations and a flat Universe had been emphasized [4]; for this reason tensor perturbations play a crucial role in discriminating between inflation and other "attractive" theories. In addition, detection of the tensor perturbations is vital to the reconstruction of the inflationary potential [5]. Denoting the spectral indices of the scalar- and tensor-metric perturbations by n and n_T respectively (scale invariance corresponds to $n-1 = n_T = 0$), and their contributions to the variance of the quadrupole CBR anisotropy by S and T, the value of the inflationary potential and its first two derivatives are given by [6]

$$\begin{aligned} V_{50}/{m_{\rm Pl}}^4 &= 1.65(1-1.4n_T)T, \\ V'_{50}/{m_{\rm Pl}}^3 &= \pm 8.3\sqrt{\frac{r}{7}}\,T, \\ V''_{50}/{m_{\rm Pl}}^2 &= 21\left[(n-1)+0.4r\right]T \end{aligned} \tag{1}$$

where $r \equiv \frac{T}{S}$. These expressions are accurate to lowest order in the deviation from scale invariance, i.e., to order $n-1$, n_T, and subscript '50' indicates the value of the scalar field when present horizon-sized fluctuations crossed outside the horizon during inflation. In addition, there is an important consistency relation: $n_T = -\frac{1}{7}r$, which is an important test of inflation.

In the foreseeable future CBR-anisotropy measurements offer the best promise to reveal the presence of tensor perturbations. Because inflationary predictions for the metric perturbations are statistical in nature and the sky is but a finite sample of the Universe, sampling variance, or cosmic variance as it has become known, provides a fundamental limit to the separation of the tensor and scalar contribution to CBR anisotropy. In this *Letter* we show that sampling variance precludes the detection of tensor perturbations if $r \lesssim 0.2$, and further, we show how a simple experiment involving measurements of CBR anisotropy on angular scales of around 3° and 0.5° can achieve this limiting sensitivity.

In the near term our ignorance of important cosmological parameters, the Hubble constant, $H_0 \equiv 100h\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, and the baryon fraction, Ω_B, as well as the ionization history of the Universe limit even further the sensitivity, as has been strongly emphasized in Ref. [8]. Since we believe that the prospects for determining these parameters by other means is good, we shall take the optimistic view that they are "known." For definiteness, we assume $h = 0.5$, $\Omega_B = 0.05$, and the standard ionization history (i.e., any injection of energy in the post-recombination Universe is not significant enough to affect the location of the last-scattering surface). In addition, we shall, for the moment, assume that n is known and equal to unity.

We begin by briefly reviewing the statistics of CBR anisotropy in general [9], and that produced by inflation in particular. CBR-temperature fluctuations are expanded in spherical harmonics,

$$\delta T(\theta,\phi) = \sum_{l,m} a_{lm} Y_{lm}(\theta,\phi). \tag{2}$$

Isotropy in the mean guarantees that $\langle a_{lm} a^*_{l'm'}\rangle = C_l \delta_{ll'}\delta_{mm'}$, where brackets indicate an ensemble average and $C_l \equiv \langle |a_{lm}|^2\rangle$ is called the angular-power spectrum. (The variance of the quadrupole anisotropy is defined to be $Q^2 \equiv 5C_2/4\pi$.) Provided that the underlying perturbations are Gaussian (as is almost certainly the case for inflation), all predictions can be derived from the angular-power spectrum. With access to only one sky, however, for each multipole moment we can only measure $2l+1$ independent multipoles, and thus, we can only estimate C_l, $C_l^{\rm estimate} = C_l^{\rm sky} \equiv \sum_m |a_{lm}|^2/(2l+1)$. The variance due to this finite sampling is

$$\langle (C_l^{\rm sky} - C_l)^2 \rangle = \frac{2}{\text{no. of samples}} = \frac{C_l^2}{l+1/2}. \tag{3}$$

This ultimate uncertainty in our knowledge of the angular-power spectrum is the key to our limit to r.

Most experiments do not directly measure the angular-power spectrum, but instead, the variance of temperature fluctuation on a given angular scale. The predicted variance is

$$\langle \delta T^2 \rangle = \sum_l \frac{2l+1}{4\pi} C_l W_l. \tag{4}$$

The window function W_l depends on the beam size and chopping strategy. Very roughly, an experiment that measures the temperature difference between directions separated by angle θ has a window function that is centered around $l \sim 180°/\theta$, with width of order l. If the experiment samples the full sky, then the variance of $\langle \delta T^2 \rangle_{\rm sky}$ is

$$\langle (\langle \delta T^2 \rangle_{\rm sky} - \langle \delta T^2 \rangle)^2 \rangle = 2 \sum_l \frac{2l+1}{(4\pi)^2} C_l^2 W_l^2, \tag{5}$$

where the brackets with subscript "sky" distinguish sky average from ensemble average.

Figure 1 shows the angular-power spectra arising from scale-invariant scalar and tensor perturbations [10]. CBR anisotropy experiments estimate the sum of the scalar and tensor contributions, $C_l = C_l^S + C_l^T$, and our problem is to separate the two contributions. (Likewise, $Q^2 = S + T$, where $S = 5C_2^S/4\pi$ and $T = 5C_2^T/4\pi$.) Differences in the metric between two points on the last-scattering surface (Sachs-Wolfe effect) are the dominant contribution to both angular-power spectra at large angles ($l \lesssim 60$). Tensor perturbations decrease in amplitude once their wavelengths become smaller than the Hubble radius which explains why the tensor spectrum drops at small angles ($l \gtrsim 60$). The scalar angular-power spectrum increases slowly with l due to the pre-recombination oscillations of the baryon-electron-photon fluid, giving rise to the "Doppler peak" at $l \sim 200$ [11].

Our strategy for separating C_l^S and C_l^T is to measure the variance in temperature fluctuations at two angular scales. The window function for one experiment, W_l^B, is centered at small angles ($l \sim 200$) where C_l^T is insignificant. The window function for the other experiment, W_l^A, is centered at intermediate angles ($l \sim 50$). In essence, experiment B measures the amplitude of the scalar spectrum, and experiment A measures "excess power" at large angles due to tensor perturbations. We define a measure of this excess power,

$$Z \equiv \frac{\langle \delta T_A^2 \rangle_{\rm sky}}{\langle \delta T_A^2 \rangle_{r=0}} - 1, \tag{6}$$

where $\langle \delta T_A^2 \rangle_{r=0} = \sum_l \frac{2l+1}{4\pi} C_l^S W_l^A$ is the variance expected in experiment A for $r = 0$ and $\langle \delta T_A^2 \rangle_{\rm sky}$ is the variance actually observed by experiment A. Note that Z is proportional to r.

Because of cosmic variance a measurement of $Z \neq 0$ does not necessarily imply $r \neq 0$. Consider the cosmic variance in Z:

$$\frac{\Delta Z^2}{\langle Z \rangle^2} \equiv \frac{\langle (Z - \langle Z \rangle)^2 \rangle}{\langle Z \rangle^2} = \frac{2 \sum_l [(2l+1)(C_l^T(r) + C_l^S)^2 (W_l^A)^2]}{(\sum_l [(2l+1) C_l^T(r) W_l^A])^2}. \tag{7}$$

We have not included the cosmic variance from experiment B because it is negligible. The ratio $Z/\Delta Z$, which is proportional to r, is a measure of signal-to-noise. We choose W_l^A to be the window function that maximizes $Z/\Delta Z$, and thereby sensitivity to r. The optimal window function is shown in Fig. 1. Its shape can be understood as a signal-to-noise weighting. For small l there is large noise (cosmic variance); for large l there is small signal (C_l^T/C_l^S is small). The "signal-to-noise" peaks at $l \sim 50$.

We have numerically calculated the probability distribution $P(Z|r)$ using a Monte-Carlo method; see Fig. 2. For $r = 0.3$ it is clearly possible to rule out the hypothesis that $r = 0$, since its probability distribution has little overlap with that for $r = 0.3$.

To address the overlap of distributions quantitatively, statisticians define "size" and "power". The power is the probability, of measuring $Z < Z_1$ given $r = 0$, and the size is the probability of measuring $Z < Z_1$ given $r = r_1$. Figure 3 illustrates the likelihood for ruling out $r = 0$ with 95 per cent confidence, given that the actual value $r \neq 0$. In other words, we have fixed the power to be 0.95, and calculated the size as a function of r. As r increases, $P(Z|r)$ overlaps less with the distribution for $r = 0$, and thus the size decreases. For $r \simeq 0.2$ there is a 95 per cent chance that one can exclude the $r = 0$ hypothesis with at least 95 per cent confidence. For the cosmologist who feels lucky, we point out that for r as small as 0.1 there is a 50 per cent chance of being able to eliminate the $r = 0$ hypothesis with at least 95 per cent confidence.

While we have chosen W_l^A to maximize $Z/\Delta Z$, the breadth of this maximum (in window-function space) is large so that even substantial changes in W_l^A do not greatly affect $Z/\Delta Z$. Nor does the optimal window function have a special shape; in fact, it is very similar to the window function for SP91 [13]. However, one might ask if a more cleverly defined observable could achieve a greater sensitivity. To the contrary, we now show that the sensitivity of the two-band measurement cannot be significantly improved upon.

To address this issue we apply a likelihood-ratio test [12] that uses all possible information, i.e., all the multipoles. Likelihood-ratio tests are designed to discriminate between two hypotheses, H_0 and H_1. Their virtue is that they are "most powerful;" that is, for fixed size such a test yields maximum power. For our purposes hypothesis H_0 is the assertion that $r = 0$, and hypothesis H_1 is the assertion that $r = r_1$.

We define the likelihood ratio,

$$\lambda \equiv \frac{P(\{a_{lm}\}|r = r_1)}{P(\{a_{lm}\}|r = 0)} = e^{-\lambda^*} \prod_{l=2}^{\infty} \left(\frac{C_l^S}{C_l^S + C_l^T(r_1)} \right)^{(2l+1)/2}, \tag{8}$$

$$\lambda^* = \sum_{l=2}^{\infty} \left[\frac{\sum_m |a_{lm}|^2}{2} \left(\frac{1}{C_l^S + C_l^T(r_1)} - \frac{1}{C_l^S} \right) \right]. \tag{9}$$

We then proceed as before: generation of the distributions $P(\lambda|r = 0)$ and $P(\lambda|r = r_1)$, followed by calculation of size for a given power. The results are shown in Fig. 3 and are nearly identical to those of the previous "Z-test."

Depending on the actual values of h, Ω_B, and n the limiting sensitivity to r can vary by a factor of about two. For example, decreasing n from 1 to 0.85 improves the sensitivity by almost a factor of two (see Fig. 3) by reducing the amplitude of the scalar multipoles for $l > 2$. Next, let us briefly address the effects of uncertainty in n. As shown in Fig. 2, it is not possible to distinguish $n = 0.94$, $r = 0$ from $n = 1.0$, $r = 0.2$. Thus, if n is known to be between 0.94 and 1, r must be greater than 0.4 to ensure detection of tensor perturbations.

A logical extension of the two-band measurement is the extraction of both r and n_T via a "three-band" measurement. An independent measure of n_T permits the testing of the consistency relation: $r = -7n_T$. However, even for

r as large as 1, so that $n_T = -1/7$, ruling out $n_T = 0$ is nearly impossible. For $r \lesssim 1$, falsification of the consistency relation requires $|n_T| \gg r/7$ [14].

Finally, while we have focussed solely upon anisotropy, polarization of the anisotropy has been suggested as another way of separating tensor and scalar perturbations. At large angles ($l \lesssim 10$), linear polarization of CBR anisotropy due to tensor perturbations is about $\sqrt{60r}$ times greater than that due to scalar perturbations [15]. By means of a likelihood-ratio test for polarization, we find a limiting sensitivity of $r \simeq 0.03$. However, as the authors of Ref. [15] point out, the measurement is difficult since the polarization-to-anisotropy ratio at large angles is less than 1 per cent for $r = 1$—and closer to 0.01% for the limiting sensitivity. Further, polarization is much more difficult to measure on large angular scales. For the foreseeable future it seems unlikely that polarization experiments can achieve higher sensitivity to r.

In summary, for the near term CBR anisotropy appears to be the most promising means of discovering inflation-produced tensor perturbations. We have shown that even with perfect knowledge of the temperature of our CBR sky, the ratio of tensor to scalar perturbations r must be greater than about 0.2 to guarantee a statistically significant detection. Of course, even the failure to detect gravity waves through CBR anisotropy at this limiting sensitivity would yield some information; namely, that $V_{50}^{1/4} \lesssim 2 \times 10^{16}\,\mathrm{GeV}$. The importance of searching for tensor perturbations should provide motivation for the next generation of satellite-based CBR anisotropy experiments to achieve the sky coverage and precision necessary.

We thank Scott Dodelson for valuable comments and the use of his scalar angular-power spectra. This work was supported in part by the DOE (at Chicago and Fermilab) and by the NASA through grant NAGW-2381 (at Fermilab).

References

[1] A. H. Guth and S.-Y. Pi, *Phys. Rev. Lett.* **49**, 1110 (1982); S. W. Hawking, *Phys. Lett. B* **115**, 295 (1982); A. A. Starobinskii, *ibid* **117**,

175 (1982); J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, *Phys. Rev. D* **28**, 697 (1983).

[2] V.A. Rubakov, M. Sazhin, and A. Veryaskin, *Phys. Lett.* **115B**, 189 (1982); R. Fabbri and M. Pollock, *Phys. Lett.* **125B**, 445 (1983); L. Abbott and M. Wise, *Nucl. Phys.* **B244**, 541 (1984).

[3] N. Kaiser et al., *Mon. Not. R. astr. Soc.* **252**, 1 (1991); M. A. Strauss et al., *Astrophys. J.* **397**, 395 (1992); A. Dekel et al., *ibid.* **412**, 1 (1993).

[4] R.H. Dicke and P.J.E. Peebles, in *General Relativity: An Einstein Centenary*, eds. S.W. Hawking and W. Israel (Cambridge Univ. Press, Cambridge, 1979), p. 505; E.R. Harrison, *Phys. Rev.* **D1**, 2726 (1970); Ya.B. Zel'dovich, *Mon. Not. R. Astr. Soc.* **160**, 1P (1972).

[5] E.J. Copeland, E.W. Kolb, A.R. Liddle, and J.E. Lidsey, *Phys. Rev. Lett.* **71**, 219 (1993); *Phys. Rev. D* **48**, 2529 (1993); M.S. Turner, *ibid*, 3503 (1993).

[6] M.S. Turner, *Phys. Rev. D* **48**, 5539 (1993).

[7] G. F. Smoot et al., *Astrophys. J.* **396**, L1 (1992); C. L. Bennett et al., astro-ph/9401012 (1994).

[8] J.R. Bond et al., *Phys. Rev. Lett.* **72**, 13 (1994).

[9] For a more complete discussion, see e.g., J.R. Bond and G. Efstathiou, *Mon. Not. R. astr. Soc.* **226**, 255 (1987).

[10] The scalar angular-power spectrum is from S. Dodelson and J.M. Jubas, *Phys. Rev. Lett.* **70**, 2224 (1993), and the tensor from S. Dodelson, L. Knox and E.W. Kolb, *Phys. Rev. Lett.* **72**, 3443 (1994).

[11] W. Hu, astro-ph/9406071 (1994).

[12] For a discussion of likelihood-ratio tests as applied to CBR anisotropy, see A.C.S. Readhead et al., *Astrophys. J.* **346**, 566 (1989).

[13] J. Schuster et al., *Astrophys. J.* **412**, L47 (1993).

[14] S. Dodelson, L. Knox and E.W. Kolb, *Phys. Rev. Lett.* **72**, 3443 (1994).

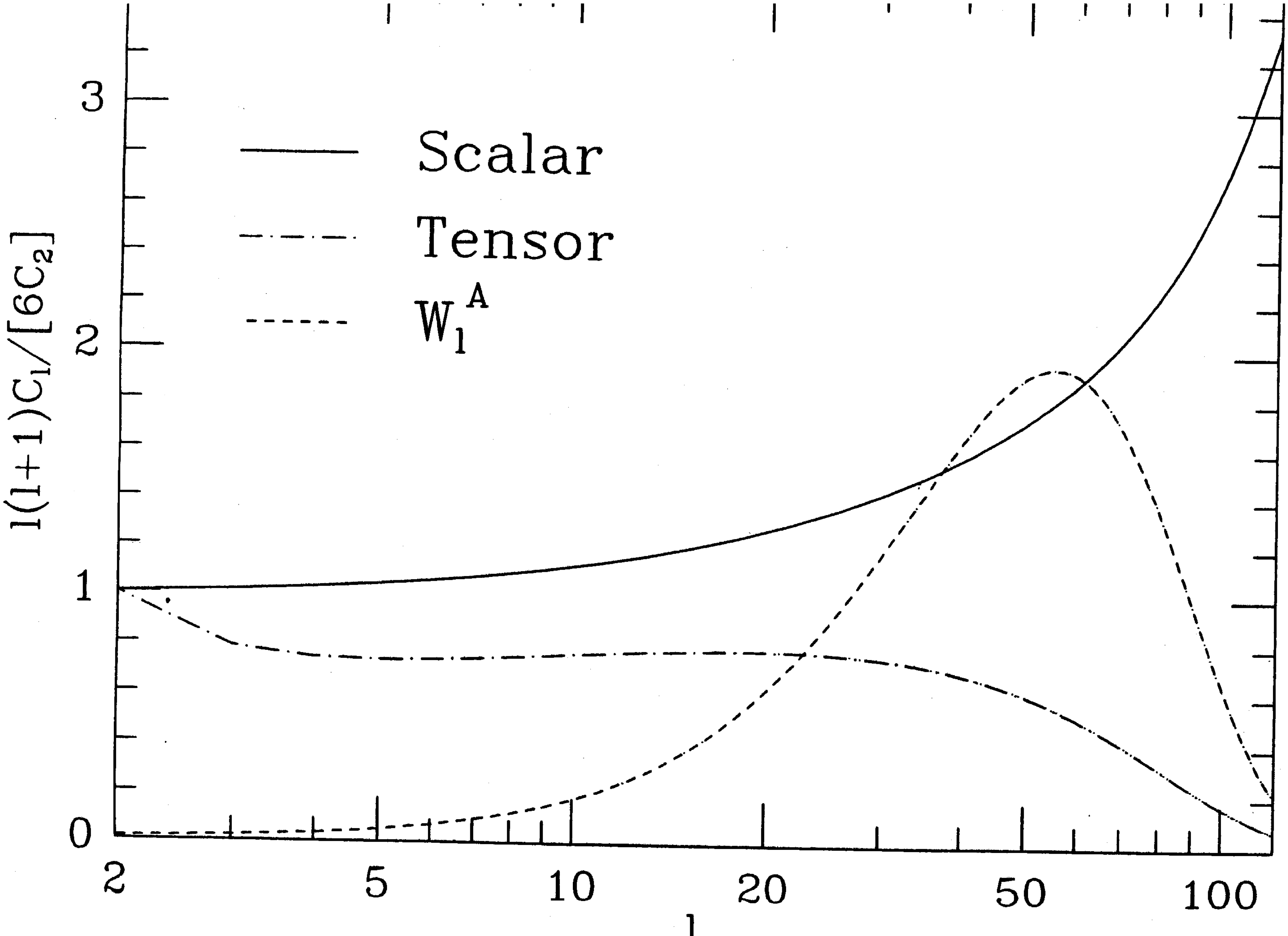
[15] R. Crittenden, R.L. Davis, and P.J. Steinhardt, *Astrophys. J.* **417**, L13 (1993).

Figure Captions

Figure 1: Angular-power spectra arising from scalar and tensor metric perturbations for $n-1=n_T=0$, $h=0.5$, and $\Omega_B=0.05$. The dashed curve is the optimal window function, W_l^A, described in the text.

Figure 2: Probability distribution $P(Z|r)$ for $(n,\, r)=(1,\, 0)$, $(1,\, 0.1)$, $(1,\, 0.2)$ and $(1,\, 0.3)$ (solid curves) and for $(n,\, r)=(0.94,\, 0)$ (broken curve).

Figure 3: Size vs. r for the "Z-test" (broken curves) and likelihood-ratio test (solid curves); $n=1$ (open symbols) and $n=0.85$ (filled symbols).



-FIG 1-

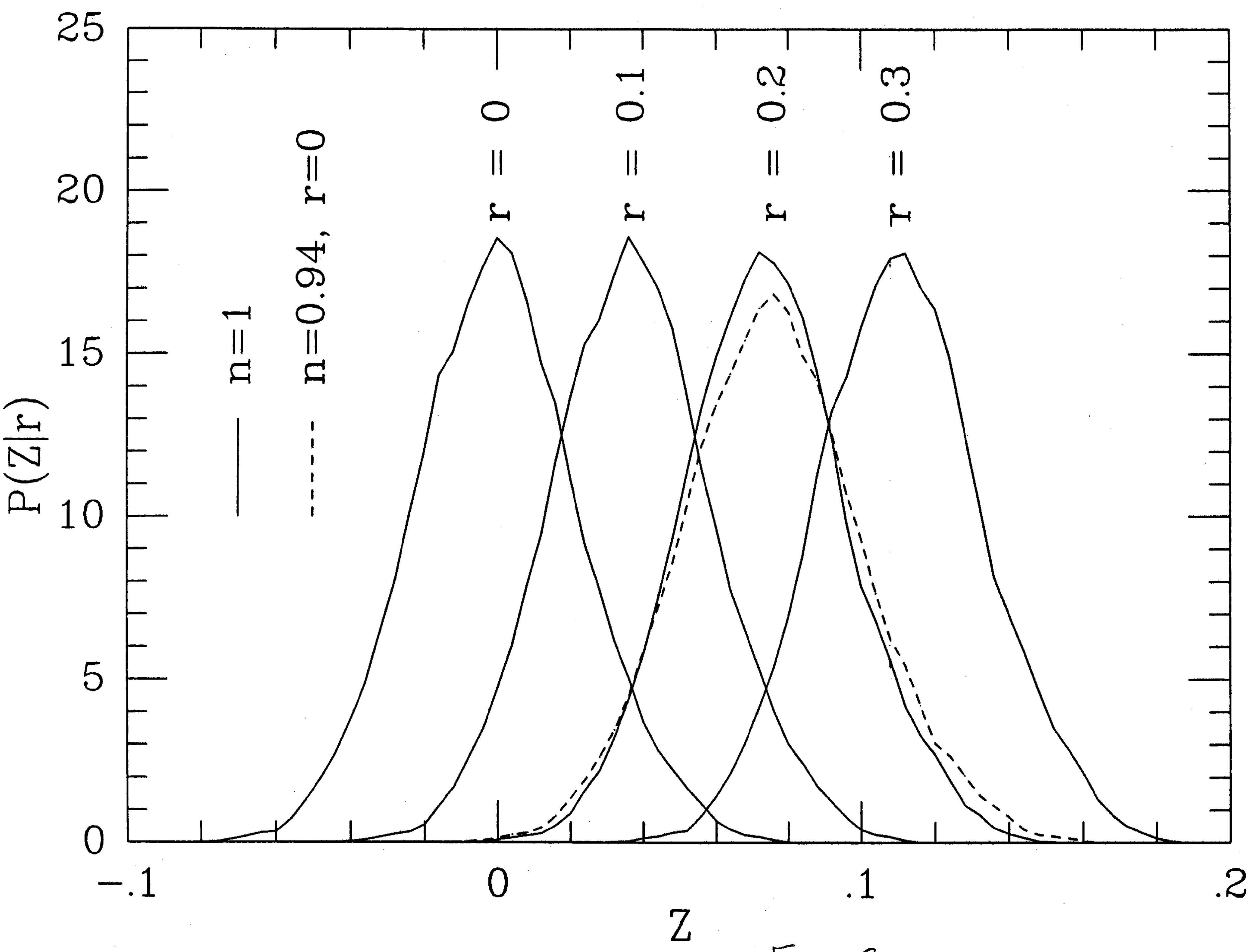
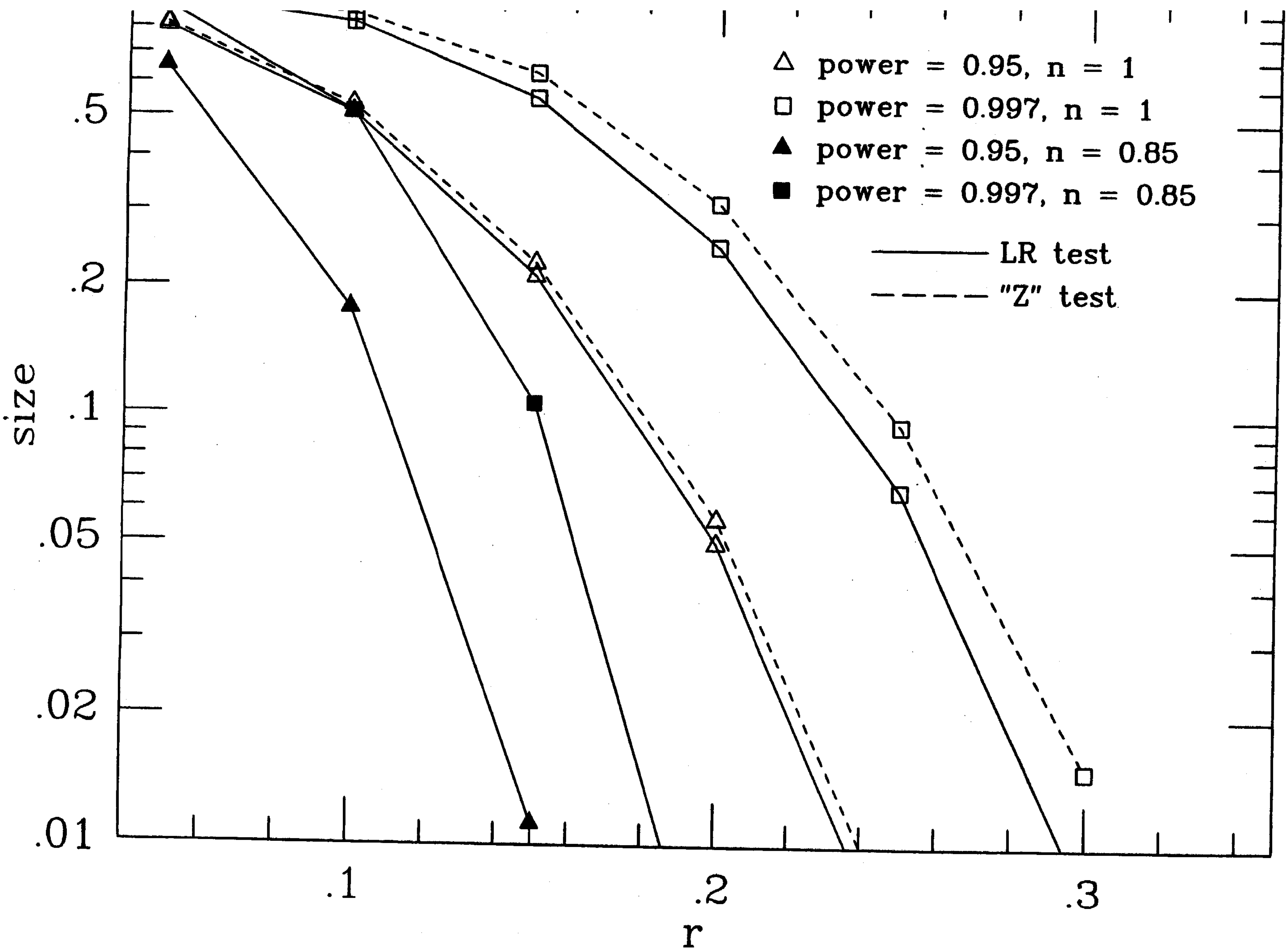


FIG. 2



size
.5
.2
.1
.05
.02
.01
.1
.2
.3
r
power = 0.95, n = 1
power = 0.997, n = 1
power = 0.95, n = 0.85
power = 0.997, n = 0.85
LR test
"Z" test

- FIG 3 -